Decomposability of Radon Measures

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Abstract

In Pedersen [6], Radon-Nikodym theorem for Radon measures is proved under the assumption that the locally compact space is σ-compact. But in fact, we can show the theorem and the duality of L^1 and L^∞ with a bit modification without σ-compactness.

1 Decomposable Measures

Def 1.1 (decomposability)**.** *Let $(X, \mathfrak{F})$ be a measurable space. A measure μ on $(X, \mathfrak{F})$ is called* ***decomposable*** *if there is a family $\{X_j\}_{j \in J}$ of disjoint measurable sets satisfying the following:*

1. $\bigsqcup_{j \in J} X_j = X$.
2. $\mu(X_j) < \infty$ *for every* $j \in J$.
3. $E \subset X$ *is measurable if and only if* $E \cap X_j$ *is measurable for all* $j \in J$.
4. $\mu(E) = \sum_{j \in J} \mu(E \cap X_j)$ *for any measurable* E *of finite measure.*

e.g.) In this paper we adopt that a **Radon measure** is a locally finite outer regular Borel measure which is inner regular for open sets. Strongly quasi-invariant measures on homogeneous spaces, containing Haar measures on locally compact groups, are decomposable because those spaces are the disjoint union of σ-compact clopen subsets $\{X_j\}_{j \in J}$ and they satisfy $\mu(E) = \sum_{j \in J} \mu(E \cap X_j)$ for any Borel subset E of finite measure by outer regularity.

Thm 1.2 (Radon-Nykodym). *Let μ be a decomposable measure on $(X, \mathfrak{F})$ with decomposition $\{X_j\}_{j\in J}$, and ν an absolutely continuous measure with respect to μ. There exists a measurable function $m : X \to [0, \infty]$ such that $\int_E m d\mu = \nu(E)$ for every E that is σ-finite for μ, and $m < \infty$ on each X_j that is σ-finite for ν.*

Proof. For every $j \in J$, pick a measurable $Y_j \subset X_j$ that is σ-finite for ν and

$$\mu(Y_j) = \sup\{\mu(E) \mid E \subset X_j : \sigma\text{-finite for } \nu\}.$$

We can see the existence of Y_j by taking a countable union of σ-finite sets for ν in X_j that approximate the right hand side. In particular, we may take $Y_j = X_j$ for all X_j that is σ-finite for ν.

By Radon-Nikodym theorem for σ-finite measures, there exists a measurable $m_j : Y_j \to [0, \infty)$ such that $\int_E m_j d\mu = \nu(E)$ for every $E \subset Y_j$. Define a function m on X as $m = m_j$ on every Y_j and $m = \infty$ otherwise. Then m is measurable because for $F \subset [0, \infty]$, $m^{-1}(F)$ is measurable $\iff$ $m_j^{-1}(F)$ and $X_j \setminus Y_j$ are measurable for all $j \in J$. By the choice of Y_j, we have $m < \infty$ on each X_j that is σ-finite for ν.

For a measurable set $E \subset X_j \setminus Y_j$, $\mu(E) = \nu(E) = 0$ or $\mu(E) > 0$ and E is not σ-finite for ν. Indeed, if E is σ-finite for ν, then $\mu(E) = \mu(Y_j \cup E) - \mu(Y_j) = 0$ because $Y_j \cup E$ is also σ-finite for ν; if E is not σ-finite for ν, then $\mu(E)$ has to be positive by absolute continuity. Thus $\int_E m d\mu = \nu(E)$ for $E \subset X_j \setminus Y_j$.

For each σ-finite E for μ, $E \cap \bigsqcup_{j \notin J_0} X_j$ is null for some at most countable set $J_0 \subset J$ by $\mu(E) = \sum_{j\in J} \mu(E \cap X_j)$. Therefore we obtain by MCT that

$$\begin{aligned}
\int_E m d\mu = \int_{E\cap \bigsqcup_{j\in J_0} X_j} m d\mu &\overset{(MCT)}{=} \sum_{j\in J_0} \int_{E\cap X_j} m d\mu \\
&= \sum_{j\in J_0} \left(\int_{E\cap Y_j} m_j d\mu + \int_{E\cap X_j \setminus Y_j} \infty d\mu \right) \\
&= \sum_{j\in J_0} \left(\nu(E \cap Y_j) + \nu(E \cap X_j \setminus Y_j) \right) = \nu(E).
\end{aligned}$$

□

You can find this theorem in Folland [2, Exercise 3.15].

Thm 1.3 (duality of L^1 and L^∞)**.** *Let* $(X, \mathfrak{F}, \mu)$ *be a decomposable measure space. Then we have an isometric isomorphism*

$$\begin{array}{cccc} I: & L^\infty(\mu) & \cong & L^1(\mu)^*. \\ & \cup & & \cup \\ & \phi & \longmapsto & \int \cdot\, \phi d\mu \end{array}$$

Proof. I is well-defined and $\|I(\phi)\| \leq \|\phi\|_\infty$ by Hölder's inequality.

(isometry) Pick a nonzero $\phi \in L^\infty(\mu)$. For each $0 < \eta < \|\phi\|_\infty$, $\{|\phi| > \eta\}$ is not null, so there is an $E \subset \{|\phi| > \eta\}$ of finite positive measure by decomposability. Then $f := (\bar{\phi}/|\phi|)\chi_E$ satisfies $\|f\|_1 = \mu(E) < \infty$ and $\int f\phi d\mu = \int_E |\phi|\, d\mu \geq \eta\mu(E)$. Thus we get $\|I(\phi)\| \geq \int f\phi d\mu / \|f\|_1 \geq \eta$, and hence $\|I(\phi)\| \geq \|\phi\|_\infty$.

(surjection) Take a decomposition $\{X_j\}_{j\in J}$ for μ and a $\varphi \in L^1(\mu)^*$. φ defines a complex finite measure ν_j on each X_j by $\nu_j(E) = \varphi(\chi_E)$, which is absolutely continuous with respect to μ. Note that the finiteness follows from $|\varphi(\chi_E)| \leq \|\varphi\|\, \mu(E)$, and σ-additivity from continuity of φ.

By Jordan decomposition and Radon-Nikodym theorem for finite measures, there exists a measurable function $\phi_j : X_j \to \mathbb{C}$ such that

$$\int_E \phi_j d\mu = \nu_j(E) = \varphi(\chi_E)$$

for any measurable $E \subset X_j$, which implies that $\int_{X_j} f\phi_j d\mu = \varphi(f)$ for every $f \in L^1(X_j, \mu)$. Here we may assume $|\phi_j| \leq \|\varphi\|$ on X_j because if $E := \{|\phi_j| > \|\varphi\|\}$ is not null, we have

$$\|\varphi\|\, \mu(E) < \int_E |\phi_j|\, d\mu = \varphi\left(\frac{\overline{\phi_j}}{|\phi_j|}\chi_E\right) \leq \|\varphi\|\, \mu(E),$$

a contradiction.

Define a function ϕ on X as $\phi = \phi_j$ on every X_j, which is measurable by decomposability of μ and bounded by $\|\varphi\|$ as shown above. For each $f \in L^1(\mu)$, since f is supported in a σ-finite set, $\{f \neq 0\} \cap \bigsqcup_{j \notin J_0} X_j$ is null for some at most countable set $J_0 \subset J$. Then it follows from $f \overset{(DCT)}{=} \sum_{j\in J_0} f\chi_{X_j}$

in $L^i(\mu)$ that

$$\begin{aligned}\varphi(f) &= \sum_{j\in J_0}\varphi(f\chi_{X_j}) = \sum_{j\in J_0}\int_{X_j} f\phi_j d\mu \\ &\overset{(DCT)}{=} \int_{\bigsqcup_{j\in J_0} X_j} f\phi d\mu = \int f\phi d\mu,\end{aligned}$$

therefore $\varphi = I(\phi)$.

□

2 Local Measurability

Def 2.1. *Let μ be a Radon measure on a locally compact space X with Borel σ-algebra $\mathfrak{B}$, and $\mathfrak{B}^1$ the subset of all Borel sets of finite measure.*

*$E \subset X$ is said to be **locally Borel** if $E \cap K$ is Borel for any $K \in \mathfrak{B}^1$. Locally Borel sets forms a σ-algebra $\mathfrak{F}$. A function on X is called **locally Borel** if its inverse image of each Borel set is locally Borel.*

*A locally Borel function is called **locally integrable** if it is integrable with respect to μ on each compact subset. This makes sense because the restriction of a locally Borel function on a compact set is a Borel function.*

*A subset E of X is called **locally null** if $E \cap K$ is null for any $K \in \mathfrak{B}^1$. A statement on X is said to hold **locally almost everywhere (l.a.e.)** if it is true except on a locally null set.*

Note that the local measurability depends on μ.

Although we defined local measurability only for Radon measures, it is in fact defined for an arbitrary measure space. If you want to know more about it, I recommend you to look up in Folland [2, Exercise 1.16, 1.22].

Recall. Let $\mathfrak{C}$ be the family of all compact sets in X. Measurable sets with respect to a Radon integral $\int = \int d\mu$ on $C_c(X)$ are defined in Pedersen [6] as follows. Define inner and outer measure of $E \subset X$ by

$$\mu_*(E) := \sup\{\mu(C) \mid \text{cpt } C \subset E\} = \sup\left\{\int_* g \,\middle|\, \chi_E \geq g \in C_c(X)_m\right\}$$

$$\mu^*(E) := \inf\{\mu(U) \mid \text{open } U \supset E\} = \inf\left\{\int^* h \,\middle|\, \chi_E \leq h \in C_c(X)^m\right\},$$

and we put

$$\begin{aligned} \mathfrak{M}^1 &= \{E \subset X \mid \mu_*(E) = \mu^*(E) < \infty\} \\ \mathfrak{M} &= \{E \subset X \mid E \cap C \in \mathfrak{M}^1 \ \forall C \in \mathfrak{C}\}. \end{aligned}$$

As is proved in Pedersen [6], $\mathfrak{M}^1 = \{E \in \mathfrak{M} \mid \mu^*(E) < \infty\}$, and μ_* and μ^* are measures on $(X, \mathfrak{M})$. It is trivial that $\mu^*|_{\mathfrak{B}} = \mu$ by outer regularity, and $\mathfrak{B}^1 = \mathfrak{B} \cap \mathfrak{M}^1$. According to locally Borel terminology, it might be good to call elements in $\mathfrak{M}$ 'locally Lebesgue.'

Lem 2.2. *The locally Borel sets ($\in \mathfrak{F}$) are contained in $\mathfrak{M}$, and hence μ_* and μ^* define measures on $(X, \mathfrak{F})$.*

Proof. Let $E \in \mathfrak{F}$, then $\mathfrak{C} \subset \mathfrak{B}^1$ implies $E \cap C \in \mathfrak{B}^1 \subset \mathfrak{M}^1$ for every $C \in \mathfrak{C}$, hence $E \in \mathfrak{M}$.

□

Lem 2.3. *i) $E \subset X$ is null if and only if $\mu^*(E) = 0$.*

ii) $E \in \mathfrak{M}$ is locally null if and only if $\mu_*(E) = 0$.

Proof.

i) If E is null, then there is a Borel set $N \supset E$ such that $\mu(N) = 0$. By outer regularity, $\mu^*(E) \leq \mu^*(N) = \mu(N) = 0$.

Conversely if $\mu^*(E) = 0$, then $\mu(N) = 0$ for some countable intersection N of open sets containing E, hence E is null.

ii) If $E \in \mathfrak{M}$ is locally null, then $C = E \cap C$ is null for any compact $C \subset E$, hence $\mu_*(E) = 0$.

Conversely if $\mu_*(E) = 0$, then we have $\mu^*(E \cap K) = \mu_*(E \cap K) = 0$ for any $K \in \mathfrak{B}^1$ by $E \cap K \in \mathfrak{M}^1$, hence $E \cap K$ is null. □

Prop 2.4. *We have for $1 \leq p < \infty$ isometric isomorphisms*

$$L^p(\mu) \cong L^p(\mu_*, \mathfrak{B}) \cong L^p(\mu^*, \mathfrak{F}) \cong L^p(\mu_*, \mathfrak{F}) \cong L^p(\mu^*, \mathfrak{M}) \cong L^p(\mu_*, \mathfrak{M}).$$

Proof. By $\mathfrak{B} \subset \mathfrak{F} \subset \mathfrak{M}$ and $\mu_* \leq \mu^*$, we have a commutative diagram of norm decreasing operators

$$\begin{array}{ccccc} L^p(\mu) & \hookrightarrow & L^p(\mu^*, \mathfrak{F}) & \hookrightarrow & L^p(\mu^*, \mathfrak{M}) \\ \downarrow & & \downarrow & & \downarrow I \\ L^p(\mu_*, \mathfrak{B}) & \hookrightarrow & L^p(\mu_*, \mathfrak{F}) & \hookrightarrow & L^p(\mu_*, \mathfrak{M}) \end{array}$$

where the horizontal arrows are isometric embeddings. It suffices to show surjective isometry for nonnegative functions. Moreover, we may assume $p = 1$ because $\|f\|_p^p = \|f^p\|_1$ for nonnegative measurable f.

(isometry) Since $\int \chi_E d\mu^* = \mu^*(E) = \mu_*(E) = \int \chi_E d\mu_*$ for any $E \in \mathfrak{M}^1$, $\int f d\mu^* = \int f d\mu_*$ holds for every $f \in L^1(\mu^*, \mathfrak{M})_+$. Thus I is an isometry, which implies that all arrows in the diagram above are isometries.

(surjection) For any $E \in \mathfrak{M}$ with $\mu_*(E) < \infty$, there is a σ-compact $F \in \mathfrak{B}^1$ such that $\mu(F) = \mu_*(E)$ i.e. $\mu_*(E \setminus F) = 0$, which shows that for any simple function $f \in L^1(\mu_*, \mathfrak{M})_+$, there is a simple function $g \in L^1(\mu)_+$ such that $f = g$ l.a.e. Therefore approximation by simple functions proves that for every $f \in L^1(\mu_*, \mathfrak{M})_+$ there exists $g \in L^1(\mu)_+$ such that $f = g$ l.a.e., hence all the spaces are isometrically isomorphic to each other.

□

Def 2.5. *By Prop(2.4), we redefine $L^p(\mu)$ as $L^p(\mu_*, \mathfrak{F})$ for $1 \leq p \leq \infty$. $L^\infty(\mu)$ is the space of l.a.e.-bounded locally Borel functions, which can be different from $L^\infty(\mu, \mathfrak{B})$.*

e.g.) Let $X = \bigsqcup_{\alpha<\omega} \mathbb{R}_\alpha$ be a disjoint union of uncountably many copies $\{\mathbb{R}_\alpha\}_{\alpha<\omega}$ of $\mathbb{R}$, where ω is the smallest uncountable ordinal. X has a Radon measure μ given by Lebesgue measure on each component.

Pick a closed $E = \{0_\alpha \in \mathbb{R}_\alpha \mid \alpha < \omega\}$. Then $\mu^*(E) = \infty$ and $\mu_*(E) = 0$, which shows that $\chi_E \neq 0$ in $L^\infty(\mu, \mathfrak{B})$ but $\chi_E = 0$ in $L^\infty(\mu_*, \mathfrak{B})$.

3 Decomposition of Radon measures

Def 3.1. *Let μ be a Radon measure on X. A family $\mathscr{F}$ of disjoint nonempty compact subsets of X is called a* ***concassage*** *(a French word meaning 'crushing') of μ if it satisfies the following:*

1. *For any open set U and $F \in \mathscr{F}$, $\mu(U \cap F) > 0$ if $U \cap F$ is nonvoid.*
2. *$F_0 := X \setminus \bigsqcup \mathscr{F}$ is locally null.*

Let's call a family $\mathscr{F}$ of disjoint nonempty compact subsets a ***preconcassage*** *(my own terminology) of μ if it has the property 1.*

Note that the empty family $\emptyset$ is a preconcassage of μ.

The definition above is according to Hewitt and Ross [5] theorem(11.39), although they did not name the family $\mathscr{F}$. The word concassage is found in Gardner and Pfeffer [3, 4].

Lem 3.2. *A preconcassage $\mathscr{F}$ of μ on X satisfies the following:*

i) $E \cap F = \emptyset$ for all but countable $F \in \mathscr{F}$ for every $E \in \mathfrak{M}^1$.

ii) $F_0 := X \setminus \bigsqcup \mathscr{F}$ is locally Borel.

Proof.

i) Given $E \in \mathfrak{M}^1$, we have an open set $U \supset E$ of finite measure. By $\sum_{F \in \mathscr{F}} \mu(U \cap F) \leq \mu(U) < \infty$ and $\mu(U \cap F) > 0$ if $U \cap F \neq \emptyset$, U intersects at most countable $F \in \mathscr{F}$, so does $E \subset U$.

ii) For any $K \in \mathfrak{B}^1$, there are by i) at most countable $F \in \mathscr{F}$ that intersects K. Thus $F_0 \cap K = K \setminus \bigsqcup_{F \in \mathscr{F}} (K \cap F)$ is a Borel set, which shows that F_0 is locally Borel.

□

Lem 3.3. *Let C be a compact subset of X. There is a compact $F \subset C$ such that $C \setminus F$ is null and $\mu(V \cap F) > 0$ for any open V that intersects F.*

Proof. Let $\mathscr{U}$ be the family of all null and relatively open subsets of C, then $U := \bigcup \mathscr{U}$ is the maximal element in $\mathscr{U}$ ($\because$ It is clear that U is relatively open and $U \in \mathfrak{B}^1$, and any compact subset of U, covered by finitely many elements in $\mathscr{U}$, is also null). Put $F := C \setminus U$: compact. Suppose $\mu(V \cap F) = 0$ for an open set V in X, then $\mu(V \cap C) = \mu(V \cap F) + \mu(V \cap U) = 0$ implies $V \cap C \subset U$ i.e. $V \cap F = \emptyset$. Thus F is what we desired.

□

Prop 3.4. *Every Radon measure μ on X admits a concassage.*

Proof. We may assume $\mu \neq 0$ since $\emptyset$ is a concassage of $\mu = 0$.

Let $\mathcal{F}$ be the set of all preconcassages of μ, ordered by inclusion. $\mathcal{F}$ is inductively ordered because every totally ordered subset $\mathcal{F}_0$ has an upper bound $\bigcup \mathcal{F}_0 \in \mathcal{F}$. Thus there is a maximal preconcassage $\mathscr{F} \in \mathcal{F}$ by Zorn's lemma.

Put $F_0 := X \setminus \bigsqcup \mathscr{F}$, which is locally Borel by Lem(3.2.ii). It suffices to show $\mu_*(F_0) = 0$ by Lem(2.3). Suppose $\mu_*(F_0) > 0$, then there is a compact

$C \subset F_0$ of positive measure. By Lem(3.3) we have a compact $F_1 \subset C$ of positive measure such that $\mu(U \cap F_1) > 0$ for any open U of nonvoid intersection with F_1. This shows that $\mathscr{F} \cup \{F_1\}$ is a preconcassage of μ, which contradicts the maximality of $\mathscr{F}$.

□

Thm 3.5. *μ_* on $(X, \mathfrak{F})$ is decomposable, and so is μ_* on $(X, \mathfrak{M})$.*

Proof. Pick a concassage $\mathscr{F}$ of μ and put $F_0 := X \setminus \bigsqcup \mathscr{F}$. We prove that $\mathscr{F} \cup \{F_0\}$ gives a decomposition of $(X, \mathfrak{F}, \mu_*)$.

1. It is evident that $F_0 \sqcup \bigsqcup \mathscr{F} = X$.
2. $0 < \mu_*(F) = \mu(F) < \infty$ for all $F \in \mathscr{F} \subset \mathfrak{C}$, and $\mu_*(F_0) = 0$ by Lem(2.3).
3. Since $\mathscr{F} \cup \{F_0\} \subset \mathfrak{F}$, $E \cap F$ and $E \cap F_0$ are locally Borel for all $F \in \mathscr{F}$ and $E \in \mathfrak{F}$.

   Conversely assume $E \subset X$ satisfies that $E \cap F_0$ and $E \cap F$ are locally Borel ($\forall F \in \mathscr{F}$). (Since $\mathscr{F} \subset \mathfrak{B}^1$, in fact $E \cap F \in \mathfrak{B}^1$ for each $F \in \mathscr{F}$.) Given $C \in \mathfrak{B}^1$ there are at most countable $F \in \mathscr{F}$ such that $F \cap C \neq \emptyset$, so it follows that

   $$E \cap C = ((E \cap F_0) \cap C) \sqcup \bigsqcup_{F \in \mathscr{F}} ((E \cap F) \cap C) \in \mathfrak{B}^1,$$

   hence $E \in \mathfrak{F}$.
4. Pick an $E \in \mathfrak{F}$ with $\mu_*(E) < \infty$. There is a σ-compact, hence Borel, subset $K \subset E$ such that $\mu(K) = \mu_*(E)$. Since $K \cap F$ is void for all but countably many $F \in \mathscr{F}$ and F_0 is locally null, we obtain

   $$\begin{aligned} \mu_*(E) = \mu(K) &= \underset{(=0)}{\mu(K \cap F_0)} + \sum_{F \in \mathscr{F}} \mu(K \cap F) \\ &\leq \underset{(=0)}{\mu_*(E \cap F_0)} + \sum_{F \in \mathscr{F}} \mu_*(E \cap F) \leq \mu_*(E) \end{aligned}$$

   i.e. $\mu_*(E) = \mu_*(E \cap F_0) + \sum_{F \in \mathscr{F}} \mu_*(E \cap F)$.

A similar proof — obtained by replacing $\mathfrak{B}^1, \mathfrak{F}$, and $C \in \mathfrak{B}^1$ by $\mathfrak{M}^1, \mathfrak{M}$, and $C \in \mathfrak{C}$ respectively — is valid for $(X, \mathfrak{M}, \mu_*)$.

□

Prop 3.6. *For any $\mathfrak{M}$-measurable function f on X, there exists a locally Borel function g such that $f = g$ l.a.e.*

Proof. Fix a $\mathfrak{M}$-measurable function f on X and a concassage $\mathscr{F}$ of μ. For each $F \in \mathscr{F}$, $f|_F$ is pointwisely approximated by $\mathfrak{M}^1$-measurable simple functions, which are equal to Borel simple functions a.e. So $f|_F$ is pointwisely approximated by Borel simple functions a.e., and hence there is a Borel function g_F on F such that $f|_F = g_F$ a.e. (i.e. $\{f|_F \neq g_F\}$ is null).

Define a function g on X by $g = g_F$ on $F \in \mathscr{F}$ and $g = 0$ on F_0, which is locally Borel because so is componentwisely. Since F_0 is locally null and any compact subset of $\{f \neq g\}$ intersects at most countable $F \in \mathscr{F}$, we get $f = g$ l.a.e.

□

Prop 3.7. *We have an isometric isomorphism*

$$L^\infty(\mu) = L^\infty(\mu_*, \mathfrak{F}) \cong L^\infty(\mu_*, \mathfrak{M}).$$

Proof. $\mathfrak{F} \subset \mathfrak{M}$ induces an isometric embedding $L^\infty(\mu) \hookrightarrow L^\infty(\mu_*, \mathfrak{M})$. Surjectivity follows from Prop(3.6), hence this is an isometric isomorphism.

□

Thm 3.8 (duality of L^1 and L^∞ for Radon measures)**.** *For a Radon measure μ on X, we have $L^\infty(\mu) \cong L^1(\mu)^*$.*

Proof. This is the direct application of the duality theorem(1.3) to $(X, \mathfrak{F}, \mu_*)$ which is found to be decomposable by Thm(3.5).

□

In order to treat several measures, we denote $\mathfrak{B}^1, \mathfrak{F}$, and $\mathfrak{M}$ for μ by $\mathfrak{B}^1_\mu, \mathfrak{F}_\mu$, and $\mathfrak{M}_\mu$ respectively.

Thm 3.9 (Radon-Nikodym for Radon measures)**.** *Let μ and ν be Radon measures on X, and ν absolutely continuous with respect to μ.*

i) Every μ-locallly null set is also ν-locally null.

ii) $\mathfrak{F}_\mu \subset \mathfrak{M}_\mu \subset \mathfrak{M}_\nu$, and hence ν_* defines absolutely continuous measures with respect to μ_* for σ-algebras $\mathfrak{F}_\mu$ and $\mathfrak{M}_\mu$.

iii) There exists a μ-locally integrable function $m : X \to [0, \infty)$ such that $\int_E m d\mu_* = \nu_*(E)$ for any $E \in \mathfrak{M}_\mu$ that is σ-finite for μ_*. m is unique up to μ-l.a.e., which is called the Radon-Nikodym derivative and denoted by $\frac{d\nu}{d\mu}$.

iv) $\int_E m d\mu^* = \nu^*(E)$ for all μ^*-σ-finite $E \in \mathfrak{M}_\mu$, containing $\mathfrak{B}^1_\mu$.

v) $\mathfrak{F}_\nu \subset \mathfrak{M}_\mu$, hence for any $f \in L^1(\nu)$, $fm \in L^1(\mu)$ is well-defined and $\int fm d\mu = \int f d\nu$.

Proof.

i) Let $E \subset X$ be a μ-locallly null set. For $K \in \mathfrak{B}^1_\nu$, we have a σ-compact $C \subset K$ with $\nu(K) = \nu(C)$. Then $\mu^*(E \cap C) = 0$ implies $\nu^*(E \cap K) \leq \nu^*(E \cap C) + \nu(K \setminus C) = 0$.

ii) For $E \in \mathfrak{M}_\mu$ and $C \in \mathfrak{C}$, there are σ-compact set $K \subset E \cap C$ and G_δ set $U \supset E \cap C$ such that $\mu(U \setminus K) = 0$, whence $\nu(U \setminus K) = 0$ shows $E \in \mathfrak{M}_\nu$.

iii.existence) Fix a concassage $\mathscr{F}$ of μ. We now know that $F_0 \in \mathfrak{F}_\mu \subset \mathfrak{M}_\nu$ is locally null not only for μ but also for ν. By Radon-Nikodym theorem(1.2), there is a μ-locally Borel function $m : X \to [0, \infty]$ such that $\int_E m d\mu_* = \nu_*(E)$ for every $E \in \mathfrak{M}_\mu$ that is σ-finite for μ_*. Since every $F \in \mathscr{F}$ and F_0 are μ_*-finite, $m < \infty$ on X. Local integrability of m comes from

$$\int_C m d\mu = \int_C m d\mu_* = \nu_*(C) = \nu(C) < \infty$$

for each $C \in \mathfrak{C}$ because μ_* and μ^* agree on $\mathfrak{M}^1_\mu$.

iii.uniqueness) Suppose m is as in the proof of existence, and m' is such another $\mathfrak{M}_\mu$-measurable function. Then $\{m \neq m'\} \in \mathfrak{M}_\mu$, and for any compact $C \subset \{m \neq m'\}$ we have

$$\int_C (m - m') d\mu = \nu(C) - \nu(C) = 0,$$

which means that $m = m'$ μ-l.a.e.

iv) It suffices to show for $E \in \mathfrak{M}^1_\mu$. In the same way as ii), we have $\nu_*(E) = \nu^*(E)$, so that

$$\int_E m d\mu^* = \int_E m d\mu_* = \nu_*(E) = \nu^*(E).$$

v) Pick $E \in \mathfrak{F}_\nu$, then it follows for any $C \in \mathfrak{C}$ that $F \cap C$ is a Borel subset of C, contained in $\mathfrak{B}^1_\mu \subset \mathfrak{M}^1_\mu$. Thus $E \in \mathfrak{M}_\mu$.

So fm is $\mathfrak{M}_\mu$-measurable and $\int fmd\mu^* = \int fd\nu^*$ holds for every ν-locally measurable simple function in $L^1(\nu)$ by iv). This shows that null functions are mapped to null functions, so $fm \in L^1(\mu)$ is well- defined, and the integral equality holds for any $f \in L^1(\nu)$ because simple functions are dense in $L^1(\nu)$. Moreover we have $\int fmd\mu = \int fd\nu$ since we can take representatives from Borel functions.

□

Prop 3.10 (chain rule)**.** *Assume that λ, μ and ν are Radon measures on X such that $\nu \ll \mu \ll \lambda$. Then*

$$\frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} = \frac{d\nu}{d\lambda} \qquad \lambda\text{-}l.a.e.$$

Proof. Now $\frac{d\nu}{d\mu}$ is measurable with respect to $\mathfrak{F}_\mu \subset \mathfrak{M}_\lambda$, and $\frac{d\mu}{d\lambda}$ is measurable with respect to $\mathfrak{F}_\lambda \subset \mathfrak{M}_\lambda$. Hence $\frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ is $\mathfrak{M}_\lambda$-measurable.

For any compact $C \subset X$, it follows from μ-local integrability of $\frac{d\nu}{d\mu}$ and Radon-Nikodym theorem(3.9.v) that

$$\nu(C) = \int_C \frac{d\nu}{d\mu} d\mu = \int_C \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} d\lambda,$$

whence

$$\nu_*(E) = \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} d\lambda_*$$

for any $E \in \mathfrak{M}_\lambda$ with $\lambda_*(E) < \infty$. Therefore $\frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} = \frac{d\nu}{d\lambda}$ λ-l.a.e. by uniqueness of the Radon-Nikodym derivative.

□

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