Derivative of an area

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1 Introduction

A **shape** has many applications in a wide range of research fields. Mathematically important are understanding shape-related quantities, such as the area of a region, the length of a curve, the (weighted) moment of a shape, etc. In this talk, a shape perturbation is performed in tha manner of Gâteaux derivative. Especially, an explicit formula for the **gradient** of an area is presented.

2 Shape representation

Let $z(\cdot)$ be a 2π-periodic parametrization of a smooth Jordan curve ∂D in $\mathbb{C}$. Since z is smooth and periodic, we have the Fourier series expansion of z.

$$z(s) = \sum_{n \in \mathbb{Z}} \hat{z}_n \varphi_n(s) \ ,$$

where $\varphi_n(s) := \frac{1}{\sqrt{2\pi}} \exp(ins)$. Note that $\{\varphi_n\}$ is a CONS of $L^2\big(S^1; \mathbb{C}\big)$.

We define by A the area of a region D whose boundary ∂D is equipped with its parametrization z.

$$A(z) := \iint_D 1 \, \mathrm{d}z_1 \, \mathrm{d}z_2 \, , \tag{1}$$

where $z = z_1 + iz_2$. We now formulate this in a manner of Fourier series. We apply the Complex Gauss-Green Theorem to (1).

$$\begin{aligned} A(z) &= \frac{i}{2} \oint_{\partial D} z \, \mathrm{d}\overline{z} \\ &= \frac{i}{2} \int_0^{2\pi} z(s) \, \overline{z'(s)} \, \mathrm{d}s \\ &= \frac{i}{2} \langle z, z' \rangle_{L^2} \ . \end{aligned} \tag{2}$$

By the way, the area $A(z)$ is a real value. Is (2) "really real"? We can easily check it by rewriting (2) with Fourier series.

$$\begin{aligned} A(z) &= \frac{i}{2}\left\langle \hat{z}, \hat{z'} \right\rangle_{\ell^2} \\ &= \frac{i}{2}\langle \hat{z}, in\hat{z} \rangle_{\ell^2} \\ &= \frac{1}{2}\langle \hat{z}, n\hat{z} \rangle_{\ell^2} \in \mathbb{R}\,. \end{aligned}$$

Furthermore, it claims that the area $A(z)$ does not depend on the Fourier coefficient $\hat{z}_0$ which is just the "center" of the region. (Note that A is in fact a signed area, which is especially pointed out by the formula $A(z(s)) = -A(z(-s))$.)

3 Shape perturbation

Since the area is written in the form of L^2 norm, we obtain a perturbation formula in a form of L^2 inner product.

$$\begin{aligned} A(z_1) - A(z_2) &= \frac{i}{2}\left[\langle z_1, z_1' \rangle_{L^2} - \langle z_2, z_2' \rangle_{L^2}\right] \\ &= \frac{i}{2}\left[\left\langle z_1 - z_2 + z_2, (z_1 - z_2 + z_2)' \right\rangle_{L^2} - \langle z_2, z_2' \rangle_{L^2}\right] \\ &= \frac{i}{2}\left[\left\langle z_1 - z_2, (z_1 - z_2)' \right\rangle + 2i\,\mathrm{Im}\langle z_1 - z_2, z_2' \rangle\right] \\ &\approx -\mathrm{Im}\langle z_1 - z_2, z_2' \rangle \qquad \text{as } \|z_1 - z_2\|_{H^1} \to 0. \end{aligned}$$

More rigorously speaking, this is a Gâteaux derivative, which is a generalized directional derivative, of the area function $A \colon H^1 \to \mathbb{R}$.

$$\begin{aligned} \mathrm{d}A(z;\Delta z) &:= \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} A(z + h\Delta z) \\ &= -\mathrm{Im}\langle \Delta z, z' \rangle_{L^2} \\ &= -\mathrm{Re}\left[-i\langle \Delta z, z' \rangle_{L^2}\right] \\ &= \langle -iz', \Delta z \rangle_{L^2(S^1;\mathbb{R}^2)}\,. \end{aligned} \tag{3}$$

Note that we here identify the complex plane $(\mathbb{C}, \mathrm{Re}\langle\cdot,\cdot\rangle_{\mathbb{C}})$ with 2-dimensional Euclidean space $\left(\mathbb{R}^2, \langle\cdot,\cdot\rangle_{\mathbb{R}^2}\right)$. We thus have the following "gradient" of the area.

$$\mathrm{Grad}_{L^2(S^1;\mathbb{R}^2)}\, A(z) = -iz' = (\text{outward normal direction of } \partial D)\,,$$

which means a perturbation to the outward normal direction causes a shape to enlarge its area. Using Fourier series, we rewrite (3) as follows.

$$\mathrm{d}A(z;\Delta z) = \langle n\hat{z}, \Delta\hat{z} \rangle_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\,.$$

4 Polar coordinates

We consider the case of Polar coordinates. Suppose z is of the following form.

$$z(s) = Z(r)\,(s) := r(s)\exp(is)\ ,$$

where r is a positive-valued 2π-periodic smooth function. Then, we get

$$\begin{aligned}
A(Z(r)) &= \frac{i}{2}\big\langle Z(r), Z(r)'\big\rangle_{L^2} \\
&= \frac{i}{2}\langle r\exp(is), r'\exp(is) + ri\exp(is)\rangle_{L^2} \\
&= \frac{1}{2}\left[i\langle r, r'\rangle_{L^2(S^1;\,\mathbb{R})} + \|r\|^2_{L^2(S^1;\,\mathbb{R})}\right] \\
&= \frac{1}{2}\left[\frac{i}{2}\int_0^{2\pi}\left(r^2\right)'\mathrm{d}s + \|r\|^2_{L^2(S^1;\,\mathbb{R})}\right] \\
&= \frac{1}{2}\|r\|^2_{L^2}\ , \\
\mathrm{d}(A\circ Z)(r;\Delta r) &= \mathrm{d}A(Z(r)\,;\mathrm{d}Z(r;\Delta r)) \\
&= \mathrm{Re}\big\langle -iZ(r)', \Delta r\exp(is)\big\rangle_{L^2(S^1;\,\mathbb{C})} \\
&= \mathrm{Re}\langle -ir' + r, \Delta r\rangle_{L^2(S^1;\,\mathbb{C})} \\
&= \langle r, \Delta r\rangle_{L^2(S^1;\,\mathbb{R})}\ , \\
\mathrm{Grad}_{L^2(S^1;\,\mathbb{R})}(A\circ Z) &= r\,.
\end{aligned}$$

The last formula indicates that a perturbation to the perpendicular direction $\Delta r \in \{r\}^{\perp}$ preserves the area A, where $(\cdot)^{\perp}$ denotes the orthogonal complement in $L^2\big(S^1;\mathbb{R}\big)$. For exmample, if r is a constant, this fact means that only 0-mode perturbation cause the area to change since $\cos(ns)$ and $\sin(ns)$ are perpendicular to the constant in L^2.

5 Application: isoperimetric problem

Let L be the arclength of a Jordan curve ∂D.

$$L(z) := \int_0^{2\pi}|\mathrm{d}z(s)| = \int_0^{2\pi}|z'(s)|\,\mathrm{d}s\,.$$

We then have the Gâteaux derivative of L explicitly as follows.

$$\begin{aligned}
\mathrm{d}L(z;\Delta z) &= \langle G(z), \Delta z\rangle_{L^2(S^1;\,\mathbb{R}^2)}\ , \qquad (4)\\
G(z) &:= \mathrm{Grad}_{L^2(S^1;\,\mathbb{R}^2)}\,L(z) = -\left(\frac{z'}{|z'|}\right)'\,.
\end{aligned}$$

Eq.(4) represents that the curvature-weighted normal direction of a boundary is the gradient of L.

We consider the isoperimetric problem which is to find a shape of the largest area whose boundary keeps a certain length. Let $z_0 \in H^1\bigl(S^1; \mathbb{C}\bigr)$ be a frozen parametrization of a certain Jordan curve ∂D_0. Suppose ∂D_0 is a solution to the isoperimetric problem. We assume mathematically that, "for any deformation of ∂D_0 with keeping its length, an area A must attain a maximum at ∂D_0."

For an arbitrary element $\Delta z \in H^1\bigl(S^1; \mathbb{C}\bigr)$, we take an arbitrary continuous map (deformation) $Z\colon (-1,1) \to H^1$ such that

$$\begin{aligned} Z(0) &= z_0\,, \\ \mathrm{d}Z(\alpha;+1) &= \Delta z\,, \\ L(Z(\alpha)) &= \mathrm{const}\,. \end{aligned}$$

Owing to $L \circ Z = \mathrm{const}$, we have

$$0 = \mathrm{d}(L\circ Z)(\alpha;+1) = \langle G(Z(\alpha)), \mathrm{d}Z(\alpha;+1)\rangle_{L^2(S^1;\mathbb{R}^2)}\ .$$

We thus get

$$\mathrm{d}Z(\alpha;+1) \in \{G(Z(\alpha))\}^{\perp}\ ,$$

where $(\cdot)^{\perp}$ denotes the orthogonal complement in $L^2\bigl(S^1;\mathbb{R}^2\bigr)$. Obviously, this is a necessary and sufficient condition for $L \circ Z = \mathrm{const}$. We especially have the following at $\alpha = 0$.

$$\Delta z \in \{G(z_0)\}^{\perp}\ .$$

Owing to the assumption, $A(Z(\alpha))$ attains its maximum at $\alpha = 0$.

$$\begin{aligned} 0 &= \mathrm{d}(A\circ Z)(0;+1) \\ &= \frac{1}{2}\langle -iz_0', \Delta z\rangle_{L^2(S^1;\mathbb{R}^2)}\ , \end{aligned}$$

from which it follows

$$\Delta z \in \{-iz_0'\}^{\perp}\ .$$

Therefore, $\{G(z_0)\}^{\perp} \subset \{-iz_0'\}^{\perp}$ holds. (This should be shown by density argument rigorously.) Noting that codim $= 1$, we conclude

$$-Ciz_0' = G(z_0) = -\left(\frac{z_0'}{|z_0'|}\right)' \qquad \text{for some constant } C \in \mathbb{R}.$$

Without loss of generality, we may assume $\hat{z}_0 = 0$, that is, the center of the region D_0 is at the origin. Taking integration and absolute value, we finally obtain $|Cz_0| = 1$. In other words, a solution ∂D_0 to the isoperimetric problem must coincide with a circle.